Spring 2022

(28) 1. Compute the derivative of the following functions. Do not simplify.

(a) $f(x) = -3x^4 + 6\sqrt{x} + 10$ $= -3x^4 + 6x^{\frac{1}{2}} + 10$

$$f'(x) = -12x^3 + 3x^{-\frac{1}{2}}$$

(b) $f(x) = (x^2 - 2x)\, ln(x^2 + 1)$

$$f'(x) = (x^2-2x)\cdot \frac{2x}{x^2+1} + \ln(x^2+1)\cdot(2x-2)$$

(c) $f(x) = \frac{e^x+1}{e^x-1}$

$$f'(x) = \frac{(e^x-1)e^x - (e^x+1)e^x}{(e^x-1)^2}$$

(d) $f(x) = \sqrt{10 + x^3}$ $= (10 + x^3)^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2}(10+x^3)^{-\frac{1}{2}} \cdot (3x^2)$$

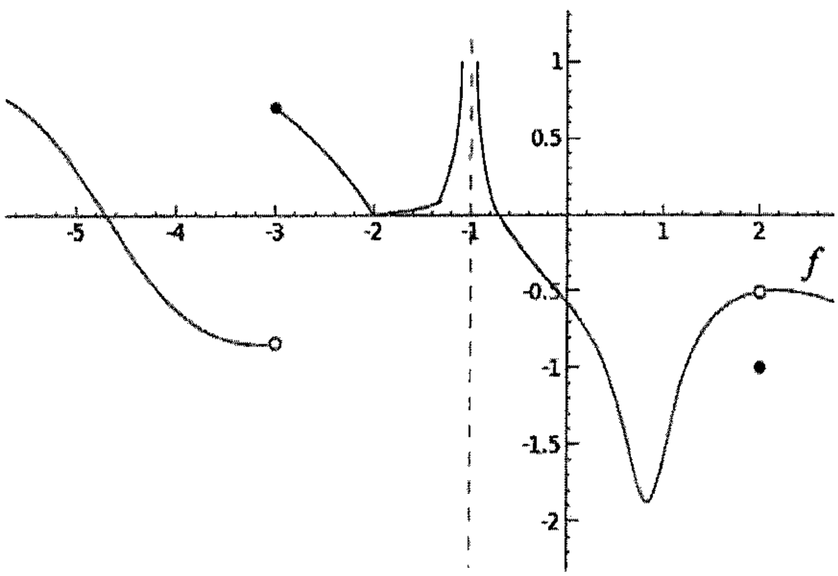
(10) 2. Calculate the following limits.

(a) $\lim\limits_{x \to -1} \frac{x^2+3x-4}{x^2-x} = \lim\limits_{x \to -1} \frac{(x+4)\cancel{(x-1)}}{x\cancel{(x-1)}} = \lim\limits_{x \to -1} \frac{x+4}{x}$

$= \frac{3}{-1} = -3$

(b) $\lim\limits_{x \to \infty} \frac{1+x-x^3}{8x^4-5} = 0$

(16) 3. Suppose that the graph of $y = f(x)$ is as given below. Use the graph to find the following limits. If a limit does not exist, write "DNE".



****There is a vertical asymptote at $x = -1$.**

(a) $\lim\limits_{x \to -3^+} f(x) =$ 0.75

(c) $\lim\limits_{x \to 2} f(x) =$ −0.5

(b) $\lim\limits_{x \to -3} f(x) =$ DNE

(d) Is f continuous at $x = -1$? NO

$f(x) = 3(x-1)^{-1}$

(12) 4. Find the equation of the tangent line to the graph of the function $f(x) = \frac{3}{x-1}$ at the point (2,3). State your answer in slope-intercept form.

$$f'(x) = -3(x-1)^{-2} \cdot 1 = \frac{-3}{(x-1)^2}$$

$$f'(2) = -3 = m$$

$$y-3 = -3(x-2)$$

$$y-3 = -3x+6$$

$$y = -3x+9$$

(12) 5. Ultra Mobile has costs that are given by

$$C(x) = 2000 + 90x + 0.2x^2$$

and price given by

$$p(x) = 210 - 0.3x$$

where x is the number of SIM cards produced. What is the number of SIM cards that Ultra Mobile must produce and sell in order to maximize profit? (Recall that the revenue is $R(x)=px$.) You must prove this using a derivative test.

$$R(x) = 210x - 0.3x^2$$

$$P(x) = R(x) - C(x) = 210x - 0.3x^2 - (2000 + 90x + 0.2x^2)$$

$$= -0.3x^2 - 0.2x^2 + 210x - 90x - 2000$$

$$= -0.5x^2 + 120x - 2000$$

$$P'(x) = -x + 120 = 0$$

$$x = 120$$

$$P''(x) = -1 < 0 \Rightarrow \max$$

120 sim cards

(20) 6. Given the function $f(x) = x^3 - 3x^2 - 9x + 5$. Use calculus to find:

a) the critical value(s)

$f' = 3x^2 - 6x - 9 = 0$

$3(x^2 - 2x - 3) = 0$

$3(x-3)(x+1) = 0$

$x = 3, -1$

b) Find where f(x) is increasing and decreasing. Use interval notation.

+ | — | +

−1 3

$f'(-2)$ $f'(0)$ $f'(5)$

incr $(-\infty, -1) \cup (3, \infty)$

decr $(-1, 3)$

c) Determine the relative extrema. Classify as max or min. Use (x,y) form.

Max $(-1, 10)$

Min $(3, -22)$

$(-1)^3 - 3(-1)^2 - 9(-1) + 5$

$-1 - 3 + 9 + 5 = 10$

$(3)^3 - 3(3)^2 - 9(3) + 5 =$

$27 - 27 - 27 + 5$

$= -22$

d) Find where f(x) is concave up and concave down. Use interval notation.

$f'' = 6x - 6 = 0$

$6x = 6$

$x = 1$

— | +

1

$f''(0)$ $f''(5)$

Con up $(1, \infty)$

Con down $(-\infty, 1)$

e) Find inflection points in (x,y) form.

IP @ $(1, -6)$

$(1)^3 - 3(1)^2 - 9(1) + 5$

$1 - 3 - 9 + 5 = -6$

(21) [redacted] 7. Compute the following integrals.

(a) $\int \left(4x^{1/2} + \frac{3}{x^2} - 12x\right) dx = \int (4x^{\frac{1}{2}} + 3x^{-2} - 12x)\, dx$

$= 4 \cdot \frac{2}{3} x^{3/2} - 3x^{-1} - 6x^2 + C$

$= \frac{8}{3} x^{3/2} - 3x^{-1} - 6x^2 + C$

(b) $\int xe^{-3x} dx$ (Use integration by parts.)

(b) $\int 5xe^{-3x^2} dx$

$u = -3x^2$

$du = -6x\, dx$

$\frac{du}{-6} = x\, dx$

$\int 5e^u \frac{du}{-6}$

$\int \frac{-5}{6} e^u du$

$= -\frac{5}{6} e^u + C$

$= \boxed{\frac{-5}{6} e^{-3x^2} + C}$

(c) $\int_2^5 \frac{2x}{\sqrt{x^2+12}} dx$

$u = x^2 + 12$

$du = 2x$

$\int u^{-1/2} du = 2u^{1/2}$

$= 2(x^2+12)^{\frac{1}{2}} \Big]_2^5 = 2(37)^{\frac{1}{2}} - 2(16)^{\frac{1}{2}}$

$= 2\sqrt{37} - 2(4)$

$= \boxed{2\sqrt{37} - 8}$

(16) 8. Given the two functions:

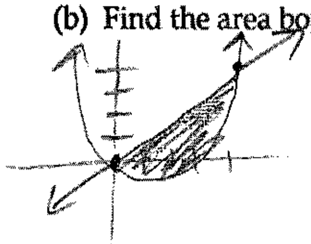
$$f(x) = x^2 - 3x \text{ and } g(x) = x$$

(a) Find the ordered pairs where f and g intersect.

$$x^2 - 3x = x$$
$$x^2 - 4x = 0$$
$$x(x-4) = 0$$
$$x = 0, 4$$

$g(0) = 0 \quad g(4) = 4 \qquad (0,0), (4,4)$

(b) Find the area bounded by the graphs of f and g. (*Hint*: Draw a sketch first.)



↖ g is on top of region

$$\text{Area} = \int_0^4 [x - (x^2 - 3x)]\,dx = \int_0^4 (-x^2 + 4x)\,dx$$
$$= -\frac{1}{3}x^3 + 2x^2 \Big]_0^4$$
$$= \frac{-64}{3} + 32 - 0$$
$$= \frac{-64 + 96}{3}$$
$$= \frac{32}{3}$$

(12) 9. According to the U.S. Census Bureau, the population of the United States can be approximated by

$$P(t) = 282.3e^{0.01t}$$

where P is in millions and t is the number of years since 2000.

Find the average value of the population from 2002 to 2006 (i.e. from $t = 2$ to $t = 6$).

$$AV = \frac{1}{6-2}\int_2^6 282.3e^{0.01t}\,dt$$

$$= \frac{1}{4}\cdot\frac{282.3}{0.01}\, e^{0.01t}\Big]_2^6$$

$$\approx 7493.91143 - 7200.07096$$

$$\approx 293.84047$$

$$\approx 294 \text{ million}$$

(15) [redacted] 10. Let

$$f(x,y) = -x^2y - 4x^4 + \frac{x}{y} \text{ [redacted]}. \quad \text{Find:} \quad \rightarrow = -x^2y - 4x^4 + xy^{-1}$$

(a) $f_y = -x^2 - xy^{-2}$

(b) $f_{yx} = -2x - y^{-2}$

(c) $f_{yx}(1,2) = -2(1) - (2)^{-2} = -2 - \frac{1}{4}$

$$= \frac{-9}{4}$$

(12) 11. ~~Find and identify the absolute minimum and maximum values of the function~~

Find where tangent line is horizontal $f(x) = x^3 - 2x^2 - 4x + 4$

~~on the interval [0, 3]. Give both coordinates.~~

$f'(x) = 3x^2 - 4x - 4 = 0$ $\quad 3x+2=0 \quad x-2=0$

$3x=-2$

$(3x+2)(x-2)=0 \quad x = -2/3, \; x=2$

$\boxed{\left(-\frac{2}{3}, \frac{148}{27}\right), \; (2, -4)}$

~~$f(2) = 8-8-8+4 = -4$ ← min~~

~~$f(0) = 4$ ← max~~

~~$f(3) = 27-18-12+4 = 1$~~

Abs. ~~max at (0,4)~~

Abs. ~~min at (2,-4)~~

$\left(-\frac{2}{3}\right)^3 - 2\left(\frac{2}{3}\right)^2 - 4\left(\frac{-2}{3}\right) + 4$

$-\frac{8}{27} - \frac{8}{9} + \frac{8}{3} + 4 = \frac{148}{27}$

(12) 12. Let

$$f(x,y) = x^3 - 3xy + y^3.$$

The critical points of $f(x, y)$ are (0, 0), (1, 1). Identify each critical point as a relative minimum, a relative maximum, or a saddle point, showing work using the D-test.

$f_x = 3x^2 - 3y \qquad f_y = -3x + 3y^2$

$f_{xx} = 6x \qquad f_y = 6y$

$f_{xy} = -3$

$(0,0)$: $D = [(6 \cdot 0) \cdot (6 \cdot 0)] - (-3)^2 = -9 < 0$

saddle at (0,0)

$(1,1)$: $D = [(6 \cdot 1) \cdot (6 \cdot 1)] - (-3)^2 = 36 - 9 = 27 > 0$

$f_{xx}(1,1) = 6 > 0 \Rightarrow$ min

Rel. min at (1,1)

(14) 13. Suppose x TV's are produced at one factory, and y TV's are produced at a second factory.

Use the method of Lagrange multipliers to find the minimum value of the company's cost function

$$C(x, y) = 6x^2 + 12y^2$$

subject to the constraint that 90 TV's are produced total, i.e. $x + y = 90$.

$C(x, y, \lambda) = 6x^2 + 12y^2 - \lambda(x + y - 90)$

$C_x = 12x - \lambda = 0 \rightarrow \lambda = 12x$

$C_y = 24y - \lambda = 0$

$C_\lambda = -(x + y - 90) = 0$

$24y - 12x = 0$

$(12)(x + y = 90)$

$-12x + 24y = 0$

$$\begin{array}{r} 12x + 12y = 1080 \\ + \quad -12x + 24y = 0 \\ \hline 36y = 1080 \end{array}$$

$y = 30$

$x + 30 = 90$

$x = 60$

$C(60, 30) = 6(60)^2 + 12(30)^2 = 32400$

$(60, 30, 32400)$