Spring 2022

(28) 1. **Compute the derivative of the following functions. Do not simplify.**

(a) $f(x) = -3x^4 + 6\sqrt{x} + 10$ $= -3x^4 + 6x^{\frac{1}{2}} + 10$

$f'(x) = -12x^3 + 3x^{-\frac{1}{2}}$

(b) $f(x) = (x^2 - 2x)\, ln(x^2 + 1)$

$f'(x) = (x^2-2x) \cdot \frac{2x}{x^2+1} + \ln(x^2+1) \cdot (2x-2)$

(c) $f(x) = \frac{e^x+1}{e^x-1}$

$f'(x) = \frac{(e^x-1)e^x - (e^x+1)e^x}{(e^x-1)^2}$

(d) $f(x) = \sqrt{10 + x^3}$ $= (10 + x^3)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(10+x^3)^{-\frac{1}{2}} \cdot (3x^2)$

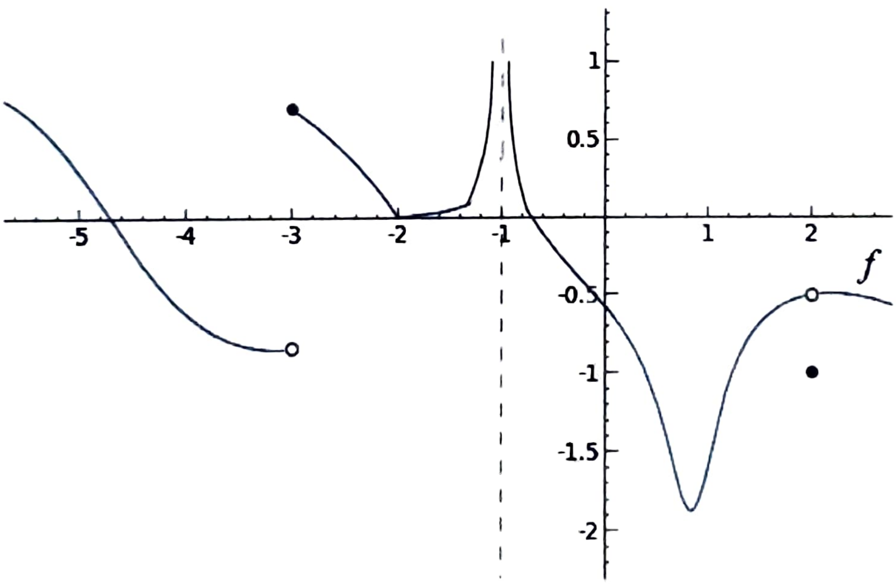
(10) 2. Calculate the following limits.

(a) $\lim_{x \to -1} \frac{x^2+3x-4}{x^2-x} = \lim_{x \to -1} \frac{(x+4)\cancel{(x-1)}}{x\cancel{(x-1)}} = \lim_{x \to -1} \frac{x+4}{x}$

$= \frac{3}{-1} = -3$

(b) $\lim_{x \to \infty} \frac{1+x-x^3}{8x^4-5} = 0$

(16) 3. Suppose that the graph of $y = f(x)$ is as given below. Use the graph to find the following limits. If a limit does not exist, write "DNE".



****There is a vertical asymptote at $x = -1$.**

(a) $\lim_{x \to -3^+} f(x) = 0.75$

(c) $\lim_{x \to 2} f(x) = -0.5$

(b) $\lim_{x \to -3} f(x) =$ DNE

(d) Is f continuous at $x = -1$? NO

$f(x) = 3(x-1)^{-1}$

(12) 4. Find the equation of the tangent line to the graph of the function $f(x) = \frac{3}{x-1}$ at the point (2,3). State your answer in slope-intercept form.

$f'(x) = -3(x-1)^{-2} \cdot 1 = \frac{-3}{(x-1)^2}$

$f'(2) = -3 = m$

$y-3 = -3(x-2)$

$y-3 = -3x+6$

$y = -3x+9$

(12) 5. Ultra Mobile has costs that are given by

$$C(x) = 2000 + 90x + 0.2x^2$$

and price given by

$$p(x) = 210 - 0.3x$$

where x is the number of SIM cards produced. What is the number of SIM cards that Ultra Mobile must produce and sell in order to maximize profit? (Recall that the revenue is $R(x)=px$.) You must prove this using a derivative test.

$R(x) = 210x - 0.3x^2$

$P(x) = R(x) - C(x) = 210x - 0.3x^2 - (2000 + 90x + 0.2x^2)$

$= -0.3x^2 - 0.2x^2 + 210x - 90x - 2000$

$= -0.5x^2 + 120x - 2000$

$P'(x) = -x + 120 = 0$

$x = 120$

$P''(x) = -1 < 0 \Rightarrow$ max

120 sim cards

(20) 6. Given the function $f(x) = \frac{3x}{x+1}$

(a) State the domain of f. $\{x \mid x \neq -1\}$ or $(-\infty,-1)\cup(-1,\infty)$

(b) Find the vertical asymptote(s).

$x=-1$

(c) Find the horizontal asymptote(s). $y=3$

(d) Give the intervals over which f is increasing.

$$f'(x) = \frac{(x+1)(3)-(3x)(1)}{(x+1)^2} = \frac{3x+3-3x}{(x+1)^2} = \frac{3}{(x+1)^2}$$

$+ \quad +$

-1

VA

$f'(-2)=3>0$

$f'(0)=3>0$

Increasing on $(-\infty,-1)\cup(-1,\infty)$

(e) Give the intervals over which f is concave up.

$f'(x) = 3(x+1)^{-2}$

$f''(x) = -6(x+1)^{-3} = \frac{-6}{(x+1)^3}$

$\uparrow \quad \downarrow$

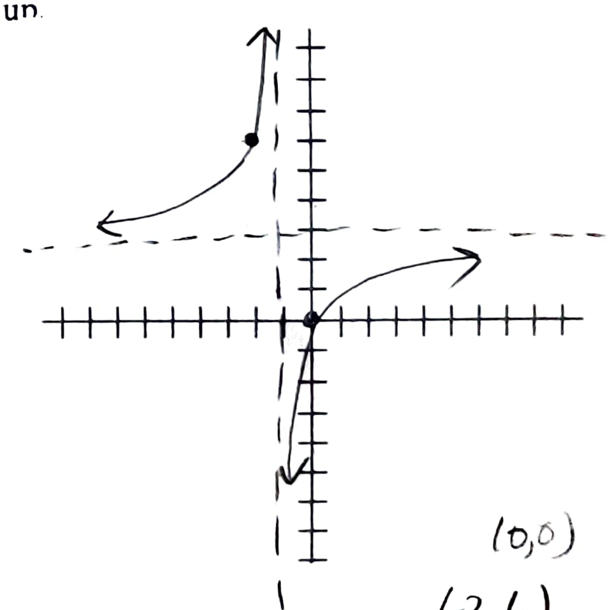
-1

VA

$f''(-2)=6>0$

$f''(0)=-6<0$

Concave up on $(-\infty,-1)$



$(0,0)$

$(-2,6)$

(f) Sketch on the above axes the graph of $f(x)$. Draw the asymptotes as dashed lines. Mark the x- and y- intercepts. Plot additional points as needed.

$f(-2) = \frac{-6}{-1} = 6$

(21) [redacted] 7. Compute the following integrals.

(a) $\int\left(4x^{1/2} + \frac{3}{x^2} - 12x\right) dx$ $= \int\left(4x^{\frac{1}{2}} + 3x^{-2} - 12x\right) dx$

$= 4 \cdot \frac{2}{3} x^{3/2} - 3x^{-1} - 6x^{2} + C$

$= \frac{8}{3} x^{3/2} - 3x^{-1} - 6x^{2} + C$

(b) $\int \overset{u}{x}\,\overset{dv}{e^{-3x} dx}$ (Use integration by parts.)

$du = dx \quad v = -\frac{1}{3}e^{-3x}$

$= -\frac{1}{3}xe^{-3x} - \int -\frac{1}{3}e^{-3x}\, dx$

$= -\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}\, dx$

$= -\frac{1}{3}xe^{-3x} + \frac{1}{3}\left(-\frac{1}{3}\right)e^{-3x} + C$

$= -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C$

(c) $\int_2^5 \frac{2x}{\sqrt{x^2+12}} dx$

$u = x^2 + 12$

$du = 2x$

$\int u^{-1/2}\, du = 2u^{1/2}$

$= 2\left(x^2+12\right)^{\frac{1}{2}}\Big]_2^5 = 2(37)^{\frac{1}{2}} - 2(16)^{\frac{1}{2}}$

$= 2\sqrt{37} - 2(4)$

$= 2\sqrt{37} - 8$

(16) 8. Given the two functions:

$f(x) = x^2 - 3x$ and $g(x) = x$

(a) Find the ordered pairs where f and g intersect.

$$x^2 - 3x = x$$
$$x^2 - 4x = 0$$
$$x(x-4) = 0$$
$$x = 0, 4$$

$g(0) = 0 \quad g(4) = 4 \qquad (0,0), (4,4)$

(b) Find the area bounded by the graphs of f and g. (*Hint*: Draw a sketch first.)

g is on top of region

$$\text{Area} = \int_0^4 [x - (x^2 - 3x)]\,dx = \int_0^4 (-x^2 + 4x)\,dx$$
$$= -\frac{1}{3}x^3 + 2x^2 \Big]_0^4$$
$$= -\frac{64}{3} + 32 - 0$$
$$= \frac{-64 + 96}{3}$$
$$= \frac{32}{3}$$

(12) 9. According to the U.S. Census Bureau, the population of the United States can be approximated by

$$P(t) = 282.3e^{0.01t}$$

where P is in millions and t is the number of years since 2000.

Find the average value of the population from 2002 to 2006 (i.e. from $t = 2$ to $t = 6$).

$$AV = \frac{1}{6-2}\int_2^6 282.3e^{0.01t}\,dt$$

$$= \frac{1}{4}\cdot\frac{282.3}{0.01}\,e^{0.01t}\Big]_2^6$$

$$\approx 7493.91143 - 7200.07096$$

$$\approx 293.84047$$

$$\approx 294 \text{ million}$$

(15) [redacted] 10. Let

$$\rightarrow = -x^2y - 4x^4 + xy^{-1}$$

$f(x,y) = -x^2y - 4x^4 + \frac{x}{y}$ [redacted]. Find:

(a) $f_y = -x^2 - xy^{-2}$

(b) $f_{yx} = -2x - y^{-2}$

(c) $f_{yx}(1,2) = -2(1) - (2)^{-2} = -2 - \frac{1}{4}$

$$= \frac{-9}{4}$$

(12) 11. Find and identify the absolute minimum and maximum values of the function

$$f(x) = x^3 - 2x^2 - 4x + 4$$

on the interval [0, 3]. Give both coordinates.

$f'(x) = 3x^2 - 4x - 4 = 0$

$(3x+2)(x-2) = 0$

$x = \cancel{-2/3}, 2$

$f(2) = 8 - 8 - 8 + 4 = -4$ ← min

$f(0) = 4$ ← max

$f(3) = 27 - 18 - 12 + 4 = 1$

Abs. max at $(0,4)$

Abs. min at $(2,-4)$

(12) 12. Let

$$f(x, y) = x^3 - 3xy + y^3.$$

The critical points of $f(x, y)$ are (0, 0), (1, 1). Identify each critical point as a relative minimum, a relative maximum, or a saddle point, showing work using the D-test.

$f_x = 3x^2 - 3y$ $\quad f_y = -3x + 3y^2$

$f_{xx} = 6x$ $\quad f_y = 6y$

$f_{xy} = -3$

$(0,0)$: $D = [(6\cdot 0)\cdot(6\cdot 0)] - (-3)^2 = -9 < 0$

saddle at $(0,0)$

$(1,1)$: $D = [(6\cdot 1)\cdot(6\cdot 1)] - (-3)^2 = 36 - 9 = 27 > 0$

$f_{xx}(1,1) = 6 > 0 \Rightarrow$ min

Rel. min at $(1,1)$

(14) 13. Suppose x TV's are produced at one factory, and y TV's are produced at a second factory.

Use the method of Lagrange multipliers to find the minimum value of the company's cost function

$$C(x, y) = 6x^2 + 12y^2$$

subject to the constraint that 90 TV's are produced total, i.e. $x + y = 90$.

$C(x, y, \lambda) = 6x^2 + 12y^2 - \lambda(x + y - 90)$

$C_x = 12x - \lambda = 0 \rightarrow \lambda = 12x$

$C_y = 24y - \lambda = 0$

$C_\lambda = -(x + y - 90) = 0$

$24y - 12x = 0$

$(12)(x + y = 90)$

$-12x + 24y = 0$

$12x + 12y = 1080$

$+ \; -12x + 24y = 0$

$36y = 1080$

$\underline{y = 30}$

$x + 30 = 90$

$\underline{x = 60}$

$C(60, 30) = 6(60)^2 + 12(30)^2 = \underline{32400}$

$(60, 30, 32400)$