Equations and their Graphs

I. LINEAR EQUATIONS

A. GRAPHS

Any equation with first powers of x and/or y is referred to as a linear equation. When graphed, all ordered (x, y) pairs that satisfy a linear equation form a straight line.

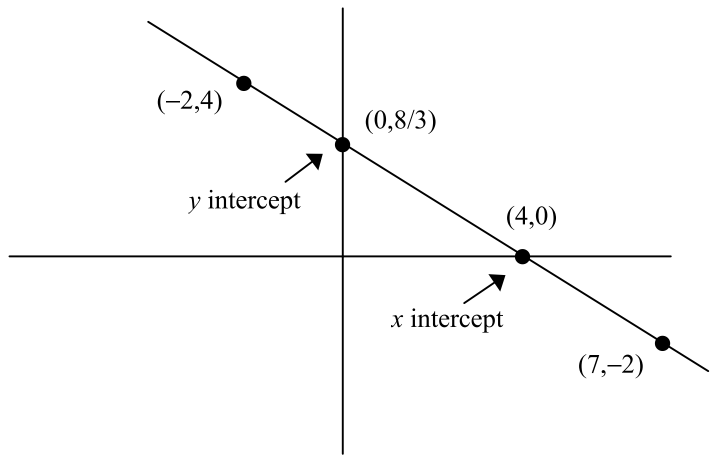
Example. Find 4 ordered pairs (including x and y intercepts) that satisfy $2x + 3y = 8$. Graph the line.

$$2(0) + 3y = 8$$
$$y = 8/3 \Rightarrow (0, 8/3)$$

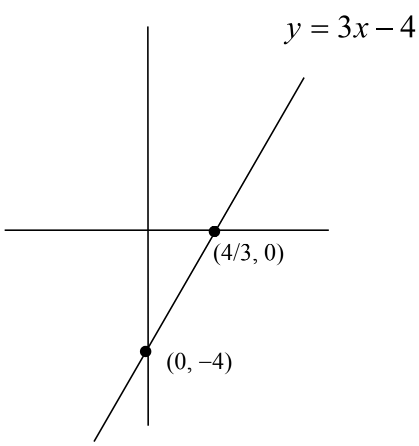
$$2(-2) + 3y = 8$$
$$y = 4 \Rightarrow (-2, 4)$$

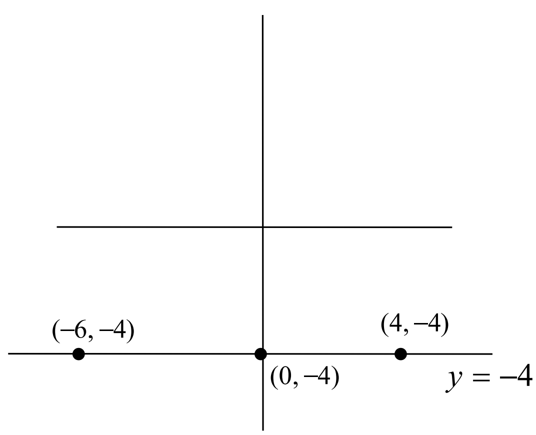
$$2x + 3(0) = 8$$
$$y = 4 \Rightarrow (4, 0)$$

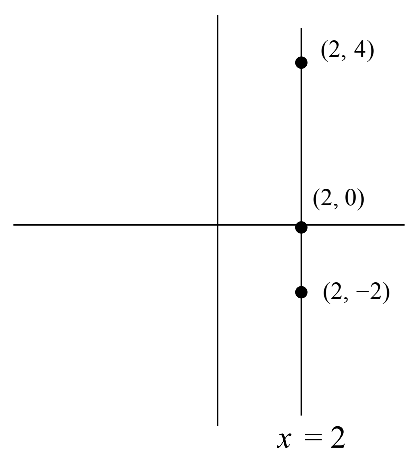
$$2x + 3(-2) = 8$$
$$x = 7 \Rightarrow (7, -2)$$



Here are 3 more examples of graphs of linear equations.







B. SLOPE

The most important characteristic of a line is the value assigned to the ratio comparing the amount of vertical change to horizontal change. Known as **slope**, this value can be defined in many ways:

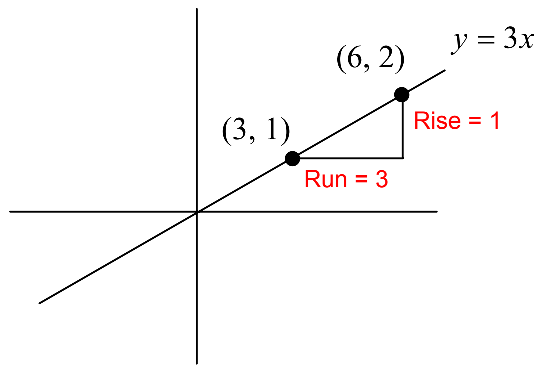
$$\text{slope or } m = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

↑ delta means change

In this course, slope will often be described as the **Rate of Change** and will (eventually) be associated with the term **Derivative**.

Example. Find the rate of change for each of the following lines.

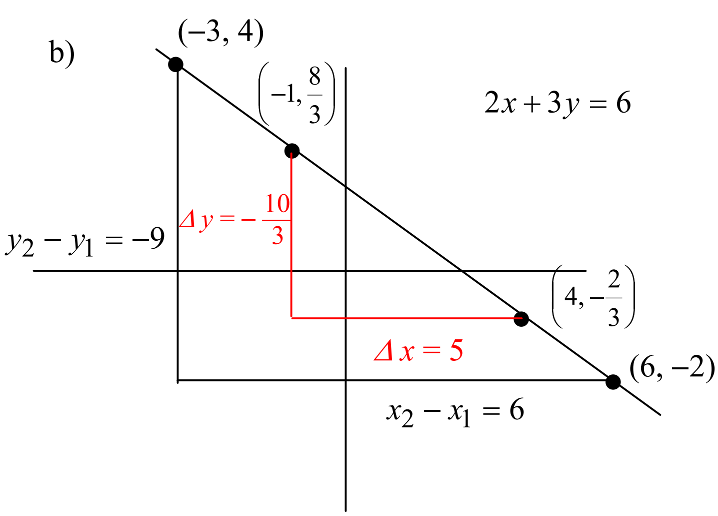
a)



$$\text{Rate of change } = \frac{1}{3}$$

$$m = \frac{1}{3}$$

b)



Using (–1, 8/3) and (4, –2/3),

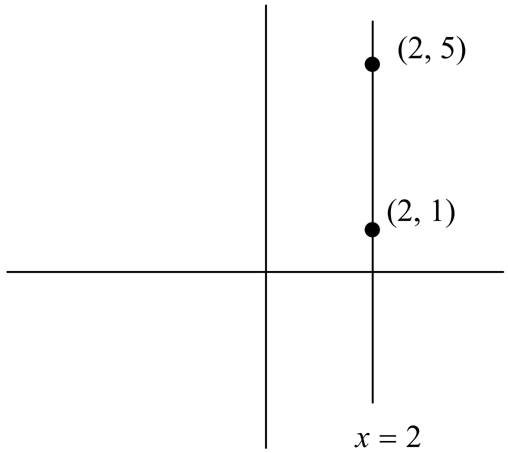
$$m = \frac{\Delta y}{\Delta x} = \frac{-\frac{10}{3}}{5} = -\frac{10}{3} \cdot \frac{1}{5} = -\frac{2}{3}\ .$$

Using (–3, 4) and (6, –2),

$$m = \frac{y_2 - y_1}{x_2 - x_1} = -\frac{6}{9} = -\frac{2}{3}\ .$$

Notice that the rate of change of a line does not depend on the points selected.

c)



$$m = \frac{5-1}{2-2} = \frac{4}{0}$$

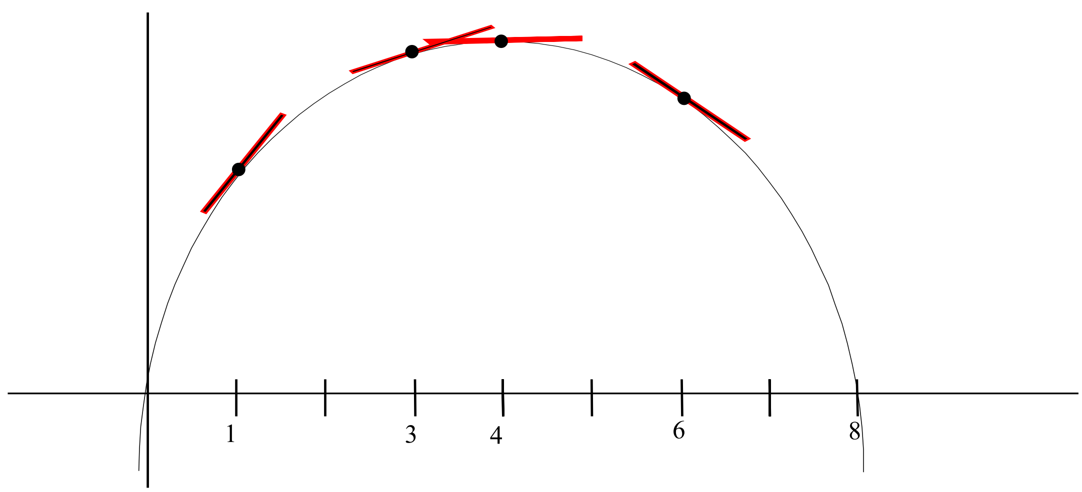
Since division by 0 is undefined, a vertical line is said to have "No Slope". Horizontal lines have slope 0.

C. TANGENT LINES

Question: Can there be a rate of change for nonlinear equations?

Answer: Yes. But in order to discuss rate of change (and still use our original concept of slope), we will look at lines that are tangent to a given curve.

Here is a curve with several tangent lines.



Even without calculating the slopes of these lines, several conclusions may be drawn.

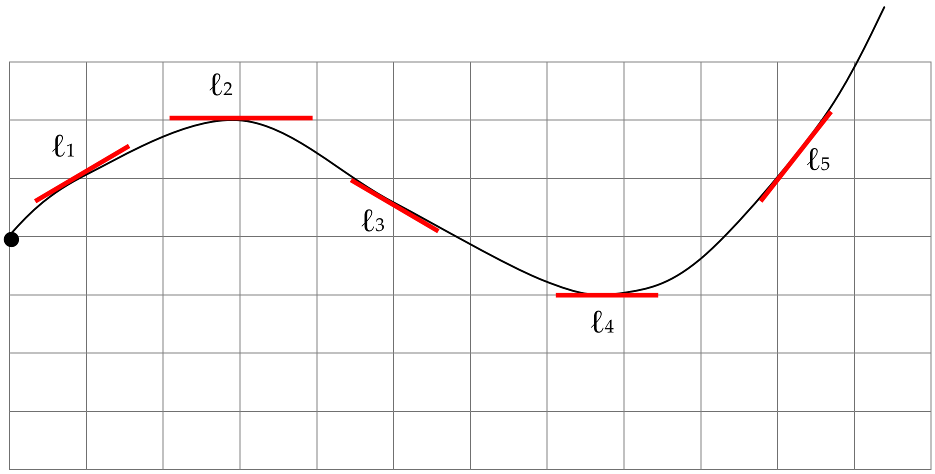
1) At $x = 1$ and 3, the tangent lines run uphill and, thus, have positive slope. The slope at $x = 1$ is greater since its tangent line is steeper.

2) The tangent line at $x = 6$ has a negative slope.

3) At $x = 4$, the tangent line appears to be horizontal and thus has slope 0. Notice that when the rate of change is 0, the graph has reached its highest point.

Exercise 1. Which of the following tangent lines have

a) $m > 0$? b) $m < 0$? c) $m = 0$?



Answers

II. QUADRATIC EQUATIONS

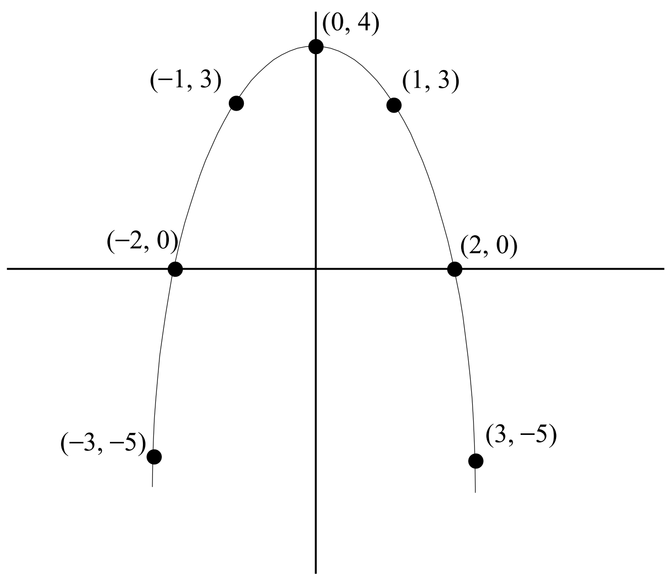
Only linear equations have graphs that result in lines. The graphs of all nonlinear equations will be "curves".

An equation in the form $y = ax^2 + bx + c$ ($a \neq 0$), is referred to as "Quadratic" and its graph is a parabola. Here are examples of the graphs of two quadratic equations along with the tables used to find points on each.

Example. $y = 4 - x^2$

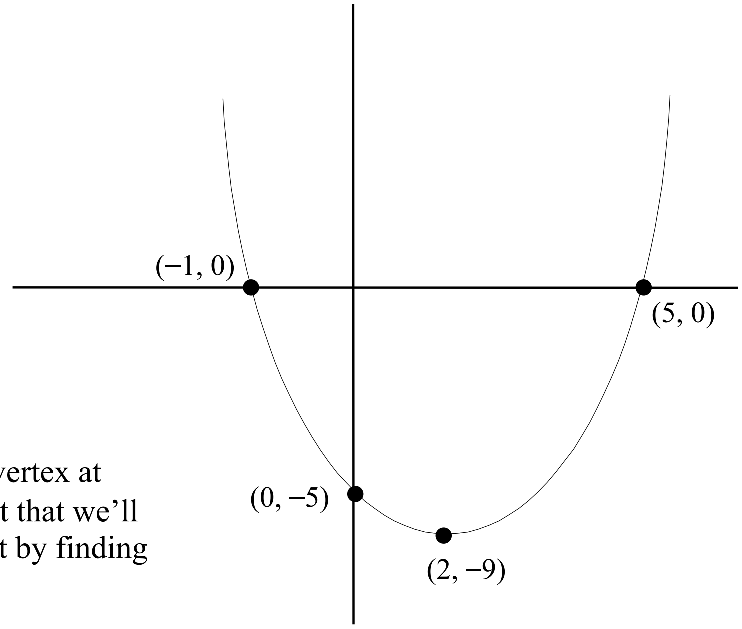
x	y
0	4
1	3
−1	3
±2	0
±3	−5

A parabola that opens down is said to be "concave down". The point (0, 4) is known as the vertex.



Example. $y = x^2 - 4x - 5$

x	y
0	−5
1	−8
3	−8
5	0
6	7
−1	0



This parabola is concave up with vertex at (2, −9). The vertex is so important that we'll give you a formula to find it. Start by finding the x value using $x = -\frac{b}{2a}$.

$$x = -\frac{b}{2a} = -\frac{(-4)}{2(1)} = 2$$

Now substitute to find y.

$$y = 2^2 - 4(2) - 5 = -9$$

We can also say that −9 is the minimum value of the equation. In our first example, 4 is the maximum value. Notice that a tangent line drawn at a minimum or maximum is horizontal.

Are you beginning to see a pattern? Horizontal tangent lines ⇒ slope 0 ⇒ maximum or minimum values ☺☺.

Exercise 2. Graph each parabola. Start by determining the vertex using $x = -\frac{b}{2a}$. Then find 2 points on each "side" of the vertex as well as x and y intercepts. State the maximum or minimum value.

a) $y = x^2 - 3$

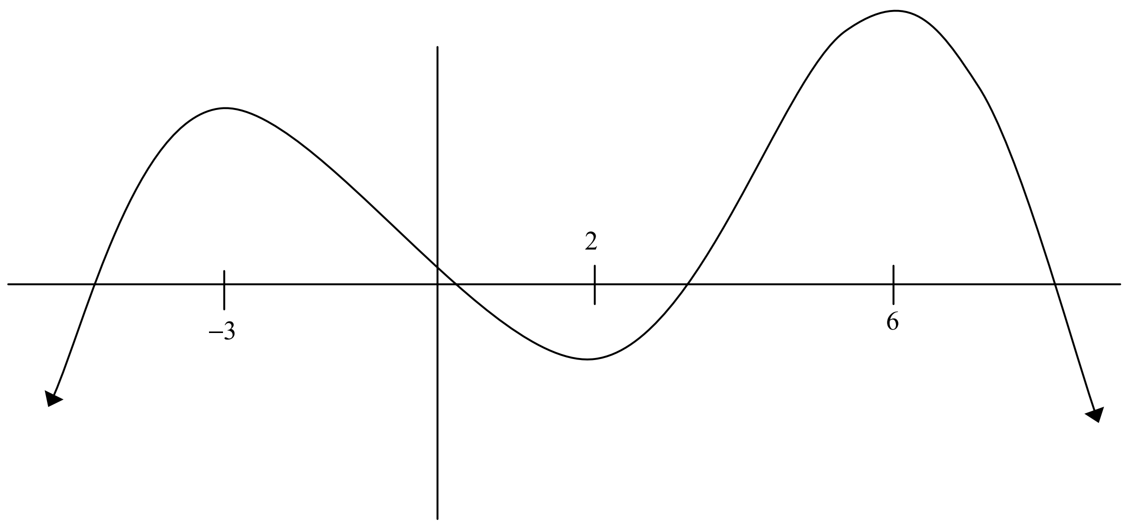
b) $y = x^2 - 4x$ Answers

c) $y = 6x - x^2$

d) $y = -x^2 + 6x + 7$ Answers

III. POLYNOMIALS

Any equation higher in degree than a quadratic (quadratics are 2nd degree) is given the name **polynomial**. As demonstrated below, the graph of a polynomial winds up and down.



Let us discuss three important characteristics of a polynomial.

1. Turning Points

   Whereas a quadratic has only one turning point (either $\cup$ or $\cap$), a polynomial may have several. Known as **extrema**, these high and low points are designated as **maximum** or **minimum**.

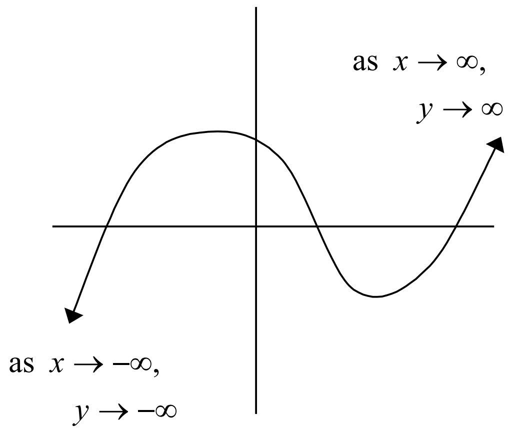
   The graph above has a maximum at $x = -3$ and 6 and a minimum at $x = 2$. Let me remind you tangent lines drawn to every extrema are horizontal which means their slope is 0.

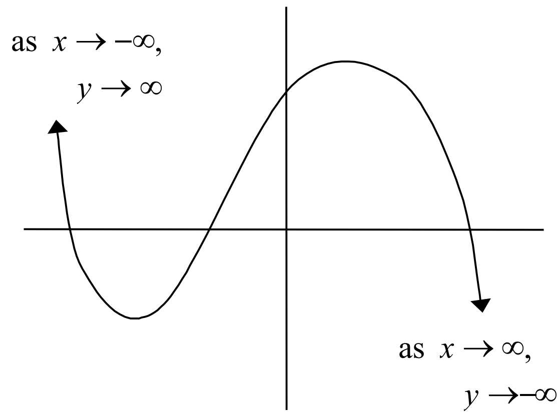
2. End Behavior

   Question: What happens to y as x gets very large?
   What happens to y as x gets very small?

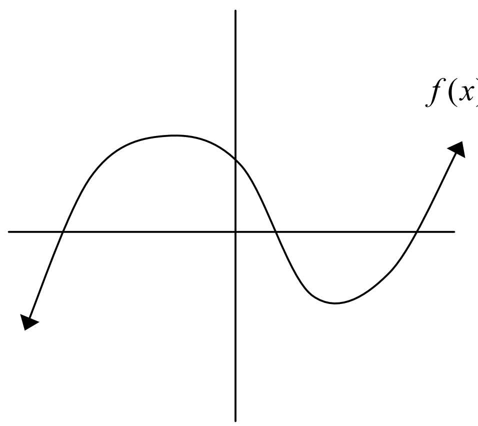
   Answering these questions will allow you to determine the behavior of the polynomial on each end.

Example. Here are two polynomials with notation describing end behavior.

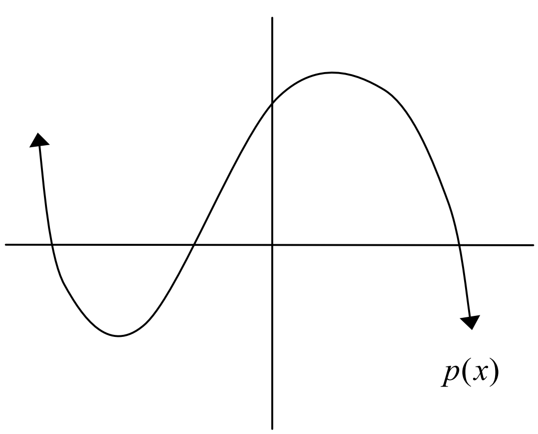




The arrow is read "approaches". In calculus, this same end behavior will be expressed using slightly different notation along with the term "limit". We simply replace the term equation with function and the letter y with $f(x)$. Here are the same graphs with their end behavior described using limit notation.



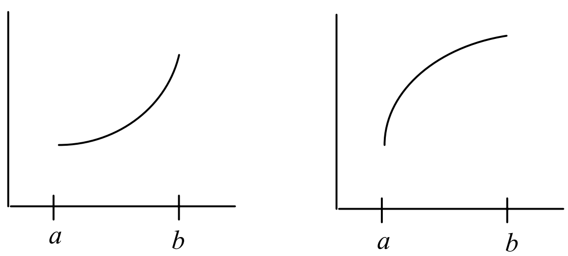
$$\underset{x\to\infty}{\text{limit}}\, f(x) = \infty$$
$$\underset{x\to-\infty}{\text{limit}}\, f(x) = -\infty$$



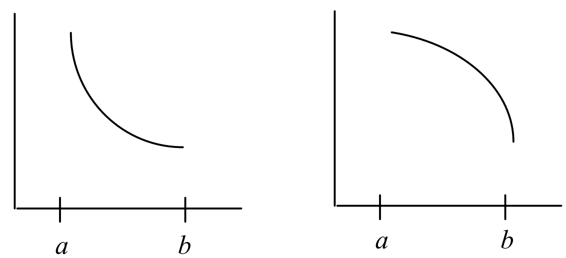
$$\underset{x\to\infty}{\text{limit}}\, p(x) = -\infty$$
$$\underset{x\to-\infty}{\text{limit}}\, p(x) = \infty$$

3. Increasing/Decreasing

Except when it remains constant (indicated by a horizontal line), an equation or function is either increasing or decreasing. Here is what we mean.



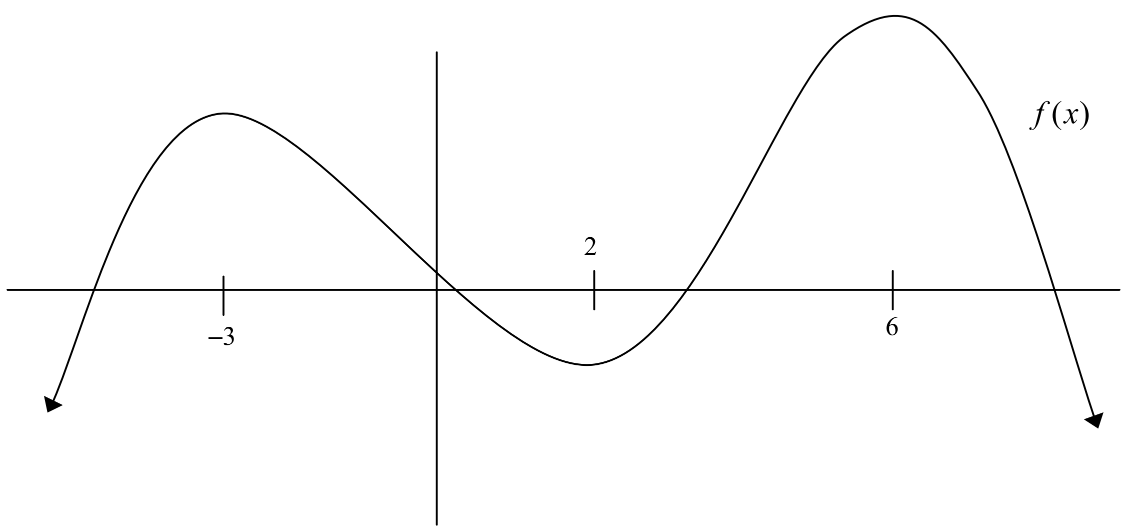
The graphs above are said to be increasing on (a, b).



The graphs above are decreasing on (a, b).

The (a, b) indicates an **interval** on x, not a point.

Let's look once again at our original graph along with all the information we've discussed.



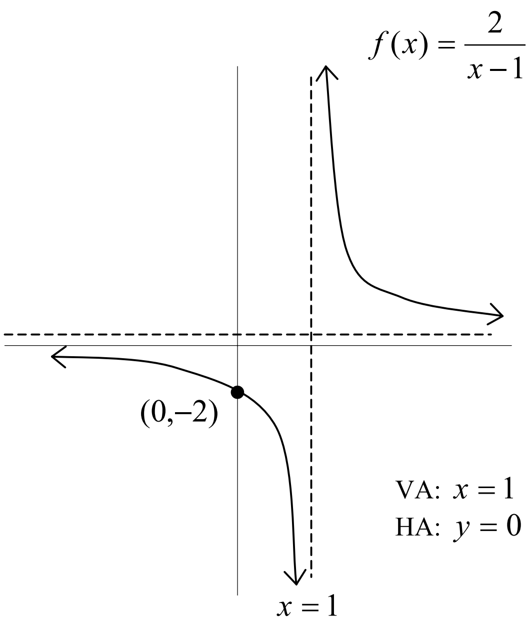
- Turning Points: Maximum at $x = -3$ and 6; Minimum at $x = 2$.
- End Behavior: $\lim\limits_{x \to -\infty} f(x) = -\infty$ $\lim\limits_{x \to \infty} f(x) = -\infty$
- Increasing/Decreasing:
  $f(x)$ is increasing on $(-\infty, -3)$ and $(2, 6)$
  $f(x)$ is decreasing on $(-3, 2)$ and $(6, \infty)$

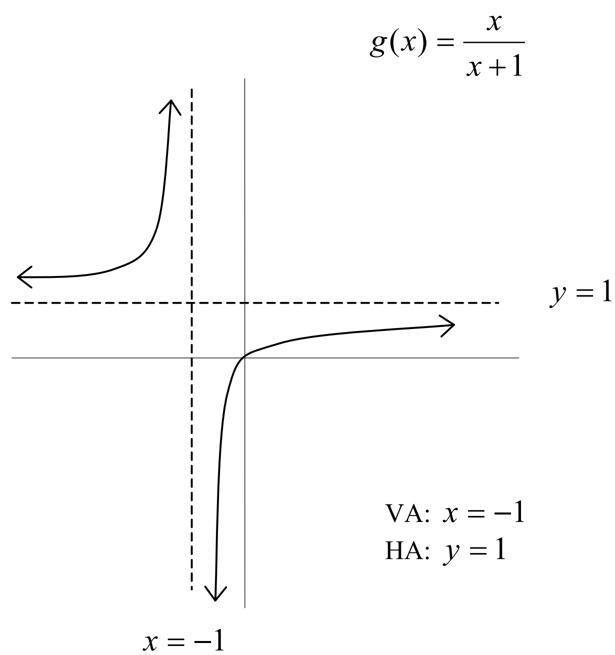
IV. RATIONAL FUNCTIONS

Don't worry that we've switched terms and are now using function instead of equation. You'll have plenty of time to adapt to function notation.

There is one other category of function that we should discuss. Rational (or fraction) functions differ significantly from polynomials. See for yourselves.

Example.



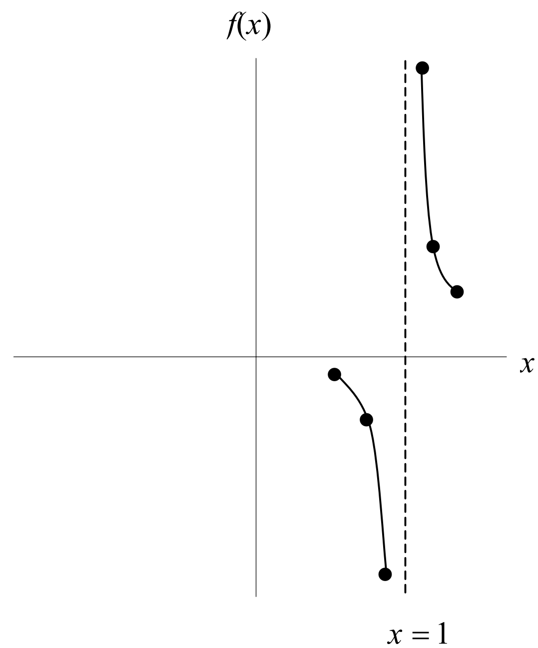


VA and HA refer to vertical and horizontal asymptotes. Asymptotes act like fences, enlosing and restraining the movement of the graph. Here's what you need to know.

A. VERTICAL AYMPTOTES

What happens with points "near" a value for which the function (equation) is undefined? Recall, division by 0 is undefined. In our first example, $f(x)$ is undefined when $x = 1$. Let us examine points near this value.

	x	$f(x) = \frac{2}{x-1}$
x approaching 1 from the left or $x \to 1^-$	.5	$f(.5) = \frac{2}{-.5} = -4$
	.9	$f(.9) = \frac{2}{-.1} = -20$
	.99	$f(.99) = \frac{2}{-.01} = -200$
	1	$f(1) = \frac{2}{0} =$ undefined $\Rightarrow$
x approaching 1 from the right or $x \to 1^+$	1.01	$\frac{2}{.01} = 200$
	1.1	$\frac{2}{.1} = 20$
	1.5	$\frac{2}{.5} = 4$



There are 2 ways to describe this behavior.

Algebra version: as $x \to 1^+$, $f(x) \to \infty$

as $x \to 1^-$, $f(x) \to -\infty$

Calculus version: $\lim\limits_{x \to 1^+} f(x) = \infty$

$\lim\limits_{x \to 1^-} f(x) = -\infty$

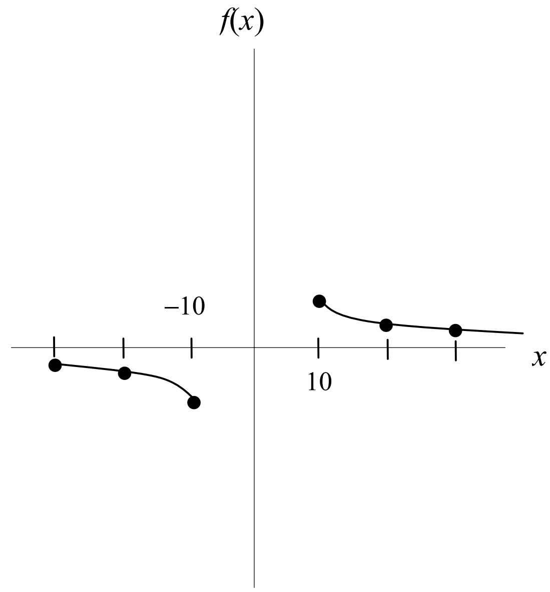
Every rational function with undefined values will have vertical asymptotes at these values. As the graph nears a vertical asymptote, it will "veer off" and head upward or downward depending on the direction of approach.

B. HORIZONTAL AYMPTOTES

To identify a horizontal asymptote we must examine end behavior. Once again a table will help.

	x	$f(x)=\frac{2}{x-1}$	
$x \to \infty$	10	$\frac{2}{9} \approx .22$	$f(x) \to 0$
	20	$\frac{2}{19} \approx .11$	
	30	$\frac{2}{29} \approx .07$	
$x \to \infty$	–10	$\frac{2}{-11} \approx -.18$	$f(x) \to 0$
	–20	$\frac{2}{-21} \approx -.09$	
	–30	$\frac{2}{-31} \approx -.06$	

$\Rightarrow$



Here's the end behavior described using limits:

$\lim_{x\to\infty} f(x) = 0$ $\qquad \lim_{x\to-\infty} f(x) = 0$ $\qquad$ conclusion $\Rightarrow$ $y = 0$ is a HA.

Can you see anything in our 2nd example, $g(x) = \frac{x}{x+1}$, that would lead you to conclude there is a HA at $y = 1$? Examine the table below.

x	$g(x)$	
10	$\frac{10}{11} \approx .91$	$\lim_{x\to\infty} g(x) = 1$
20	$\frac{20}{21} \approx .95$	
30	$\frac{30}{31} \approx .97$	
–10	$\frac{-10}{-9} \approx 1.11$	$\lim_{x\to-\infty} g(x) = 1$
–20	$\frac{-20}{-19} \approx 1.05$	
–30	$\frac{-30}{-29} \approx 1.03$	

conclusion

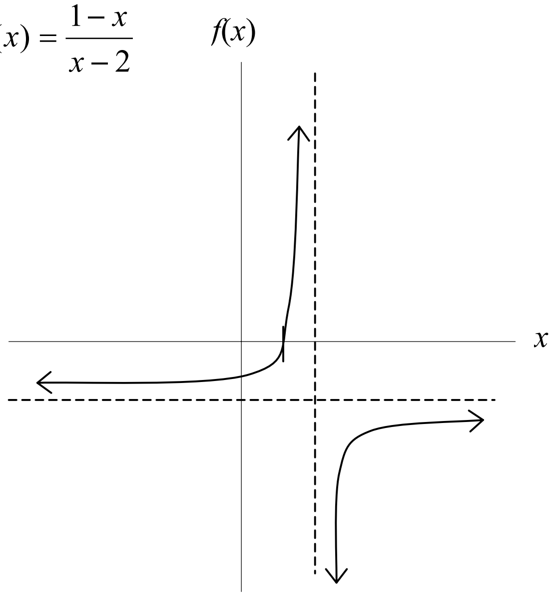
$\Rightarrow$ HA at $y = 1$

> To find Horizontal Aymptotes examine end behavior, i.e., find the limit as $x \to \infty$ or

Rather than relying on tables, we plan to show you an algebra trick to help you analyze end behavior. That will come later in the course.

Exercise 3. Use the graph below to indicate the following:

$$f(x) = \frac{1-x}{x-2}$$



a) horizontal and vertical asymptotes

b) $\underset{x \to 2^+}{\text{limit}}\ f(x) =$

c) $\underset{x \to 2^-}{\text{limit}}\ f(x) =$

d) $\underset{x \to -\infty}{\text{limit}}\ f(x) =$

e) $\underset{x \to \infty}{\text{limit}}\ f(x) =$

Answers

Answers to Exercises for Equations and their Graphs

Exercise 1.

a) $m > 0$ for ℓ_1 and ℓ_5

b) $m < 0$ for ℓ_3

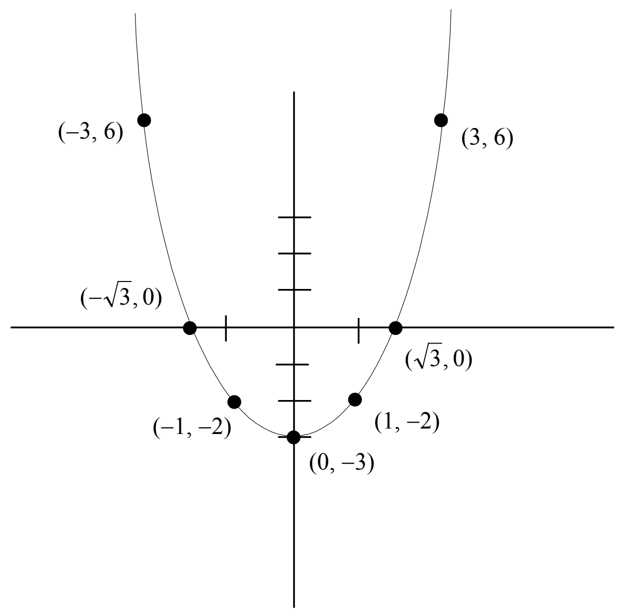
c) $m = 0$ for ℓ_2 and ℓ_4

Exercise 2.

a) $y = x^2 - 3$

x	y	
0	–3	vertex and y-intercept
±1	–2	
±3	6	
$* \pm\sqrt{3}$	0	x-intercept

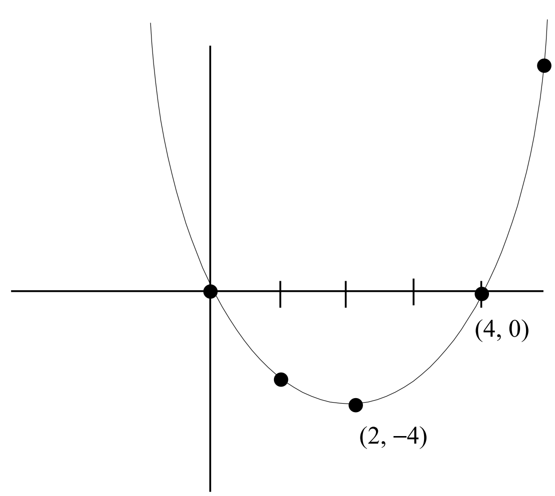
*To find x intercepts, solve $x^2 - 3 = 0$



b) $y = x^2 - 4x$

$$x = \frac{-(-4)}{2(1)} = 2$$

x	y	
2	–4	vertex
1	–3	
0	0	intercepts
4	0	
5	5	

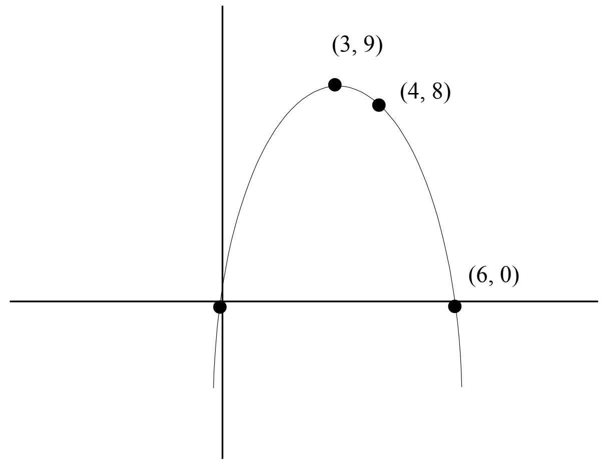


Exercise 2. (continued)

c) $y = 6x - x^2$

$$x = \frac{-6}{2(-1)} = 3$$

x	y	
3	9	vertex
4	8	
6	0	intercepts
0	0	
1	5	

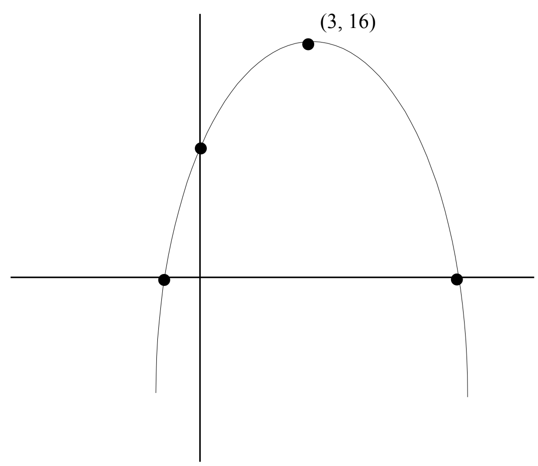


This parabola is concave down and has a maximum value of 9 when $x = 3$.

d) $y = -x^2 + 6x + 7$

$$x = -\frac{6}{2(-1)} = 3$$

x	y	
3	$-9+18+7=16$	vertex
0	7	y-int
7	0	x-int
-1	0	
6	7	



To find x-intercept, solve

$$-x^2 + 6x + 7 = 0$$
$$x^2 - 6x - 7 = 0$$
$$(x-7)(x+1) = 0$$
$$x = 7, \; x = -1$$

Exercise 3.

VA: $x = 2$ $f(x)$ is undefined at $x = 2$

HA: $y = -1$ $f(10)$ and $f(-10)$ are both ≈ -1.

This also means that $\lim_{x \to \pm\infty} f(x) = -1$.