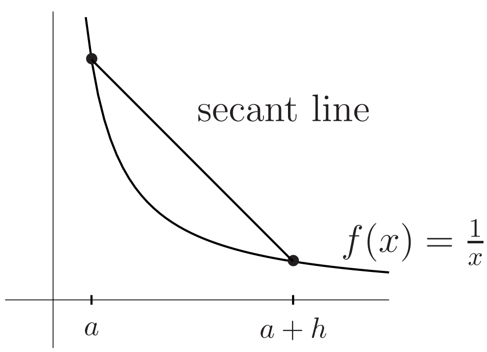
Math 250 Skills Assessment Test

The purpose of this test is purely diagnostic (before beginning your review, it will be helpful to assess both strengths and weaknesses). The test problems cover Calculus I concepts that are essential for second semester calculus. Answers are provided, and each answer has references to relevant review topics either in Math 250 or earlier courses. If anything is unclear, the review material should help.

You may click on the blue words if you wish to jump to an answer or the review topics.

If you would like to print the Skills Assessment so you can work it out on paper, please click Print.

1.



Find the slope of the secant line for $f(x) = \dfrac{1}{x}$ between the two points whose x-coordinates are a and $(a + h)$. Answer

2. Consider a circle of radius 2 centered at the origin. Find the area of a sector of the circle determined by an angle $\theta = \dfrac{\pi}{6}$ radians. Answer

3. Evaluate the following limits.

   a) $\displaystyle\lim_{x \to 2} \frac{x^2 - 4}{2x^2 + 6x - 20}$ Answer

   b) $\displaystyle\lim_{x \to 1^+} \ln(x - 1)$ Answer

c) $\lim_{x\to\infty} \dfrac{2x+1}{\sqrt{x^2+3}}$ Answer

d) $\lim_{x\to 0} \dfrac{\sin 4x}{x}$ Answer

4. Find y' for the following functions.

a) $y = 2e^x + \pi \tan x - 7\cos x$ Answer

b) $y = 3\sqrt{x} + \dfrac{7}{x} - \sin x + 3\cot x$ Answer

c) $y = \dfrac{4}{3}\ln x - \sec x - 2\csc x$ Answer

d) $y = (x^2+2x-1)(e^x + \tan x)$ Answer

e) $y = (4x^{2/3} - \sqrt[3]{x})(x^3+3x-\ln x)$ Answer

f) $y = \dfrac{x^2-2x+1}{\tan x + \cot x}$ Answer

g) $y = \dfrac{xe^x + \cos x}{\sqrt[4]{x} - 4}$ Answer

h) $y = (x^2+3)^{1/2} - \sin(2x) + e^{\tan x}$ Answer

i) $y = \sin^2 x + \sin x^2$ Answer

j) $y = \csc\sqrt{x^2+1}$ Answer

k) $x^2y + xy^2 = 4$ Answer

l) $e^{xy} = \ln(x^2 - y)$ Answer

m) $y = x^{\cos x}$ Answer

5. Find the equation of the tangent line to the curve $y = 1 + \ln x$ where $y = 1 + \ln x$ crosses the x-axis. Answer

6. Evaluate:

a) $\int x^2 dx$ Answer

b) $\int_0^{\ln 2} e^x dx$ Answer

c) $\int_0^{\pi/4} \cos x \, dx$ Answer

d) $\int_{\pi/6}^{\pi/3} \tan^2 x \, dx$ Answer

e) $\int (e^x + \cos x) dx$ Answer

f) $\int_1^{\pi/2} (\csc x \cot x + \sqrt[3]{x}) dx$ Answer

g) $\int_{-2}^{1} (x^3 + 1)^2 dx$ Answer

h) $\int x(x^2 + 1)^{1/2} dx$ Answer

i) $\int \frac{dx}{x + 1}$ Answer

j) $\int_0^{\pi/4} \sec^2 x e^{-\tan x} dx$ Answer

k) $\int e^{\pi x} dx$ Answer

l) $\int_{-1}^{0} x\sqrt{x + 1} dx$ Answer

m) $\displaystyle\int \frac{x+3}{x^2+6x-1}dx$ Answer

7. Find the area bounded by the curves $y = \tan x$, $x = \dfrac{\pi}{6}$, $x = \dfrac{\pi}{4}$, and $y = -1$. Answer

8. Find the area bounded by the curves $y = x$ and $y = 4x^3$. Answer

ANSWERS to SKILLS ASSESSMENT

1. To find the slope, we need two points. If $x = a$ then $y = f(a)$, and so one point is $(a, f(a))$. The other point is $(a + h, f(a + h))$. Then

$$\text{slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}.$$

(This quotient has a special form and is called the difference quotient.)

In this problem $f(x) = \frac{1}{x}$, or $f(\quad) = \frac{1}{(\quad)}$. Thus,

$$\begin{aligned}\text{slope} &= \frac{f(a+h) - f(a)}{h} = \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \frac{\frac{a-(a+h)}{a(a+h)}}{h} \\ &= \frac{-h}{ah(a+h)} = \frac{-1}{a(a+h)}.\end{aligned}$$

Return to Problem

(see Math 250, Review Topic 1 for help.)

2. The area A of a sector of a circle of radius r subtended by a central angle θ is given by the formula $A = \frac{1}{2}r^2\theta$. Thus,

$$A = \frac{1}{2}(2^2)\frac{\pi}{6} = \frac{\pi}{3}.$$

Return to Problem

(see Math 250, Review Topic 6 for help.)

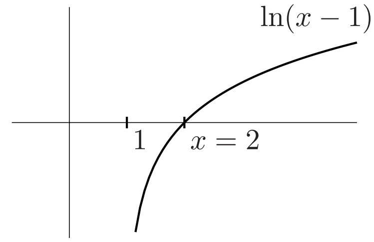
3 a) Find $\lim_{x\to 2} \frac{x^2 - 4}{2x^2 + 6x - 20}$. This has form $\frac{0}{0}$ at $x = 2$, and so we must manipulate. Let's try factoring.

$$\lim_{x\to 2} \frac{x^2 - 4}{2(x^2 + 3x - 10)} = \lim_{x\to 2} \frac{(x-2)(x+2)}{2(x-2)(x+5)} = \lim_{x\to 2} \frac{x+2}{2(x+5)} = \frac{4}{14} = \frac{2}{7}$$

Return to Problem

(see Math 250, Review Topic 11B for help.)

3. b) Find $\lim_{x\to 1^+} \ln(x-1)$. The graph of $\ln(x-1)$ has the following appearance, which is a shift of $\ln x$ one unit to the right. Thus, $\lim_{x\to 1^+} \ln(x-1) = -\infty$.



Return to Problem

(see Math 250, Review Topic 11A for help.)

3. c) Find $\lim_{x\to\infty} \dfrac{2x+1}{\sqrt{x^2+3}}$. Let's factor out the highest power of x that we can in the numerator and also the denominator.

$$\lim_{x\to\infty} \frac{2x+1}{\sqrt{x^2+3}} = \lim_{x\to\infty} \frac{x\left(2+\dfrac{1}{x}\right)}{\sqrt{x^2\left(1+\dfrac{3}{x^2}\right)}} = \lim_{x\to\infty} \frac{x\left(2+\dfrac{1}{x}\right)}{x\sqrt{1+\dfrac{3}{x^2}}}$$

$$= \lim_{x\to\infty} \frac{\left(2+\dfrac{1}{x}\right)}{\sqrt{1+\dfrac{3}{x^2}}}.$$

As $x \to \infty$, $\dfrac{1}{x} \to 0$ and $\dfrac{3}{x^2} \to 0$. Thus,

$$\lim_{x\to\infty} \frac{2x+1}{\sqrt{x^2+3}} = \frac{2}{\sqrt{1}} = 2.$$

Return to Problem

(see Math 250, Review Topic 11C for help.)

3. d) $\lim_{x\to 0} \frac{\sin 4x}{x} = \lim_{x\to 0} \frac{4}{4} \cdot \frac{\sin 4x}{x} = \lim_{x\to 0} 4 \cdot \frac{\sin 4x}{4x}$. We know $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (See Example 11B.7). This really says $\lim_{\theta\to 0} \frac{\sin(\theta)}{(\theta)} = 1$, where θ is the same for the numerator and the denominator. Thus,

$$\lim_{x\to 0} \frac{\sin 4x}{4x} = 1.$$

This means

$$\lim_{x\to 0} \frac{\sin 4x}{x} = \lim_{x\to 0} 4 \cdot \frac{\sin 4x}{4x} = 4 \cdot 1 = 4.$$

Return to Problem

(see Math 250, Review Topic 11B for help.)

4. a) $y' = 2e^x + \pi \sec^2 x + 7 \sin x$

Return to Problem

(see Math 250, Review Topic 12B for help.)

4. b) Rewrite y as $y = 3x^{1/2} + 7x^{-1} - \sin x + 3 \cot x$

$$y' = 3\left(\frac{1}{2}x^{-1/2}\right) - 7x^{-2} - \cos x - 3\csc^2 x$$

Return to Problem

(see Math 250, Review Topic 12B for help.)

4. c) $y' = \frac{4}{3} \cdot \frac{1}{x} - \sec x \tan x + 2 \csc x \cot x$

Return to Problem

(see Math 250, Review Topic 12B for help.)

4. d) $$y' = \left[\frac{d}{dx}(x^2+2x-1)\right](e^x+\tan x) + (x^2+2x-1)\frac{d}{dx}(e^x+\tan x)$$
$$= (2x+2)(e^x+\tan x) + (x^2+2x-1)(e^x+\sec^2 x)$$

Return to Problem

(see Math 250, Review Topic 12C for help.)

4. e) $$y' = \left(4\left(\frac{2}{3}\right)x^{-1/3} - \frac{1}{3}x^{-2/3}\right)(x^3+3x-\ln x) + (4x^{2/3}-x^{1/3})(3x^2+3-\frac{1}{x})$$

Return to Problem

(see Math 250, Review Topic 12C for help.)

4. f) $y' =$

$$= \frac{\left[\frac{d}{dx}(x^2-2x+1)\right](\tan x+\cot x) - (x^2-2x+1)\frac{d}{dx}(\tan x+\cot x)}{(\tan x+\cot x)^2}$$

$$= \frac{(2x-2)(\tan x+\cot x) - (x^2-2x+1)(\sec^2 x-\csc^2 x)}{(\tan x+\cot x)^2}$$

Return to Problem

(see Math 250, Review Topic 12D for help.)

4. g) $$y' = \frac{(e^x+xe^x-\sin x)(x^{1/4}-4) - (xe^x+\cos x)(\frac{1}{4}x^{-3/4})}{(x^{1/4}-4)^2}$$

Return to Problem

(see Math 250, Review Topic 12D for help.)

4. h) $y' = \frac{1}{2}(x^2+3)^{-1/2}(2x) - (\cos 2x)\cdot 2 + e^{\tan x}\sec^2 x$

Return to Problem

(see Math 250, Review Topic 12E for help.)

4. i) Rewrite $y = (\sin x)^2 + \sin(x^2)$.

$$y' = 2\sin x\cos x + \cos(x^2)\cdot 2x.$$

Return to Problem

(see Math 250, Review Topic 12E for help.)

4. j) $y' = \left[-\csc(x^2+1)^{1/2}\cot(x^2+1)^{1/2}\right]\left(\frac{1}{2}\right)(x^2+1)^{-1/2}\cdot 2x$

Return to Problem

(see Math 250, Review Topic 12E for help.)

4. k) Use implicit differentiation.

$$\begin{aligned} 2xy + x^2y' + y^2 + x(2yy') &= 0 \\ y'(x^2+2xy) &= -2xy - y^2 \\ y' &= \frac{-2xy-y^2}{x^2+2xy} \end{aligned}$$

Return to Problem

(see Math 250, Review Topic 12F for help.)

4. l) Use implicit differentiation.

$$\begin{aligned} e^{xy}[y + xy'] &= \frac{1}{x^2 - y}(2x - y') \\ e^{xy}xy' + \frac{1}{x^2 - y}y' &= \frac{2x}{x^2 - y} - ye^{xy} \\ y' &= \frac{\dfrac{2x}{x^2 - y} - ye^{xy}}{\dfrac{1}{x^2 - y} + xe^{xy}} \end{aligned}$$

Return to Problem

(see Math 250, Review Topic 12F for help.)

4. m) Take the natural log of both sides to arrive at

$$\ln y = \ln x^{\cos x} = \cos x \ln x.$$

Now use implicit differentiation

$$\frac{1}{y}y' = -\sin x \ln x + (\cos x) \cdot \frac{1}{x}\,.$$

Then

$$\begin{aligned} y' &= y\left(-\sin x \ln x + \frac{\cos x}{x}\right) \\ &= x^{\cos x}\left[-\sin x \ln x + \frac{\cos x}{x}\right] \end{aligned}$$

Return to Problem

(see Math 250, Review Topic 12G for help.)

5. To find the equation of the tangent line we need the point and slope. The curve crosses the x-axis at $f = 0 = 1 + \ln x \Rightarrow \ln x = -1 \Rightarrow x = e^{-1}$. The point is $(e^{-1}, 0)$. To find the slope we need to take the derivative and evaluate the derivative at the point. Thus $f' = \dfrac{1}{x}$ $f'(e^{-1}) = \dfrac{1}{e^{-1}} = e$. Then $y - 0 = e(x - e^{-1}) \Rightarrow y = ex - 1$.

Return to Problem

(see Math 250, Review Topic 12H for help.)

6. a) $\dfrac{x^3}{3} + C$

Return to Problem

(see Math 250, Review Topic 13A for help.)

6. b) $e^x \Big|_0^{\ln 2} = e^{\ln 2} - e^0 = 2 - 1 = 1$

Return to Problem

(see Math 250, Review Topic 13A for help.)

6. c) $\sin x \Big|_0^{\pi/4} = \sin\dfrac{\pi}{4} - \sin 0 = \dfrac{\sqrt{2}}{2}$

Return to Problem

(see Math 250, Review Topic 13A for help.)

6. d)
$$\int_{\pi/6}^{\pi/3} \tan^2 x\,dx = \int_{\pi/6}^{\pi/3} (\sec^2 x - 1)dx = (\tan x - x)\Big|_{\pi/6}^{\pi/3}$$
$$= \tan\frac{\pi}{3} - \frac{\pi}{3} - \left(\tan\frac{\pi}{6} - \frac{\pi}{6}\right) = \sqrt{3} - \frac{1}{\sqrt{3}} - \frac{\pi}{6}$$

Return to Problem

(see Math 250, Review Topic 13B for help.)

6. e) $e^x + \sin x + C$

Return to Problem

(see Math 250, Review Topic 13A for help.)

6. f) Rewrite integral as $\displaystyle\int_1^{\pi/2} (\csc x \cot x + x^{1/3})dx$

$$= \left(-\csc x + \frac{x^{4/3}}{\frac{4}{3}} \right) \Bigg|_1^{\pi/2} = -\csc\frac{\pi}{2} + \frac{3}{4}\left(\frac{\pi}{2}\right)^{4/3} - \left(-\csc 1 + \frac{3}{4} \right)$$

$$= -1 + \frac{3}{4}\left(\frac{\pi}{2}\right)^{4/3} + \csc 1 - \frac{3}{4} = -\frac{7}{4} + \frac{3}{4}\left(\frac{\pi}{2}\right)^{4/3} + \csc 1$$

Return to Problem

(see Math 250, Review Topic 13B for help.)

6. g) $\displaystyle\int_{-2}^{1} (x^6 + 2x^3 + 1)dx = \left(\frac{x^7}{7} + \frac{2x^4}{4} + x \right) \Bigg|_{-2}^{1}$

$$= \frac{(1)^7}{7} + \frac{1}{2} + 1 - \left[\frac{(-2)^7}{7} + \frac{1}{2}(-2)^4 + (-2) \right] = \frac{23}{14} + \frac{172}{14}$$

$$= \frac{195}{14}$$

Return to Problem

(see Math 250, Review Topic 13B for help.)

6. h) Let $u = x^2 + 1$, $du = 2x\,dx$.

$$\int (\underbrace{x^2+1}_{u})^{1/2} \underbrace{x\,dx}_{\frac{du}{2}} = \int \frac{1}{2} u^{1/2} du$$

$$= \frac{1}{2} \frac{u^{3/2}}{\frac{3}{2}} + C$$

$$= \frac{1}{3}(x^2+1)^{3/2} + C$$

Return to Problem

(see Math 250, Review Topic 13C for help.)

6. i) $u = x + 1$, $du = dx$.

$$\int \frac{\overbrace{dx}^{du}}{\underbrace{(x+1)}_{u}} = \int \frac{du}{u} = \ln|u| + C = \ln(x+1) + C$$

Return to Problem

(see Math 250, Review Topic 13C for help.)

6. j) $u = -\tan x$, $du = -\sec^2 x\,dx$. We now need to change the limits of integration; when $x = 0$, $u = -\tan 0 = 0$; when $x = \dfrac{\pi}{4}$, $u = -\tan\dfrac{\pi}{4} = -1$.

$$\int_{0 \,(u=0)}^{\pi/4 \,(u=-1)} e^{\overbrace{-\tan x}^{u}} \underbrace{\sec^2 x\,dx}_{-du} = \int_0^{-1} (-e^u du)$$

$$= -e^u \Big|_0^{-1} = -e^{-1} - (-e^0) = 1 - e^{-1}$$

Return to Problem

(see Math 250, Review Topic 13C for help.)

6. k) $u = \pi x$, $du = \pi\, dx$, $\displaystyle\int e^{\overbrace{\pi x}^{u}} \underbrace{dx}_{\frac{du}{\pi}} = \frac{1}{\pi}\int e^u du$

$$= \frac{1}{\pi}e^u + C = \frac{e^{\pi x}}{\pi} + C$$

Return to Problem

(see Math 250, Review Topic 13C for help.)

6. l) $u = x + 1, \quad du = dx$

$$x = -1 \quad u = -1 + 1 = 0$$

$$x = 0 \quad u = 0 + 1 = 1$$

$$\int_{\underset{u=0\nearrow}{-1}}^{\overset{u=1\searrow}{0}} \underbrace{x}_{\substack{\downarrow \\ u-1=x}} (\underbrace{x+1}_{u})^{1/2} \underbrace{dx}_{du} = \int_0^1 (u-1)u^{1/2} du$$

$$= \int_0^1 (u^{3/2} - u^{1/2}) du = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\Big|_0^1$$

$$= \frac{2}{5}\cdot 1 - \frac{2}{3}\cdot 1 - \left(\frac{2}{5}\cdot 0 - \frac{2}{3}\cdot 0\right)$$

$$= \frac{-4}{15}$$

Return to Problem

(see Math 250, Review Topic 13C for help.)

6. m) $u = x^2 + 6x - 1,\ du = (2x+6)dx = 2(x+3)dx$

$$\int \frac{\overbrace{(x+3)dx}^{\frac{du}{2}}}{\underbrace{x^2+6x-1}_{u}} = \int \frac{\frac{du}{2}}{u} = \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+6x-1| + C$$

Return to Problem

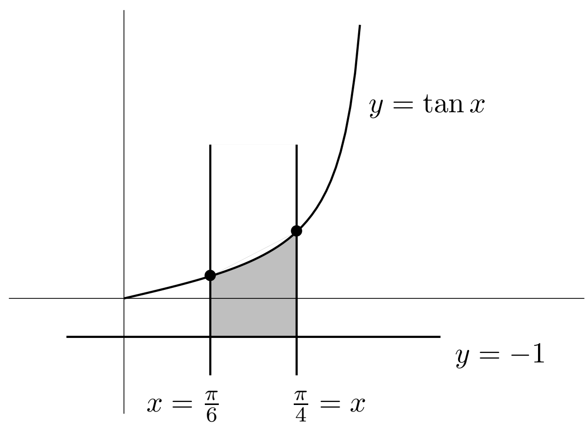
(see Math 250, Review Topic 13C for help.)

7. $$\text{Area} = \int_{\pi/6}^{\pi/4} (\tan x - (-1))dx = \ln|\sec x| + x\Big|_{\pi/6}^{\pi/4}$$

$$= \ln\left(\sec\frac{\pi}{4}\right) + \frac{\pi}{4} - \left(\ln\left(\sec\frac{\pi}{6}\right) + \frac{\pi}{6}\right)$$

$$= \ln\sqrt{2} - \ln\frac{2}{\sqrt{3}} - \frac{\pi}{12}$$

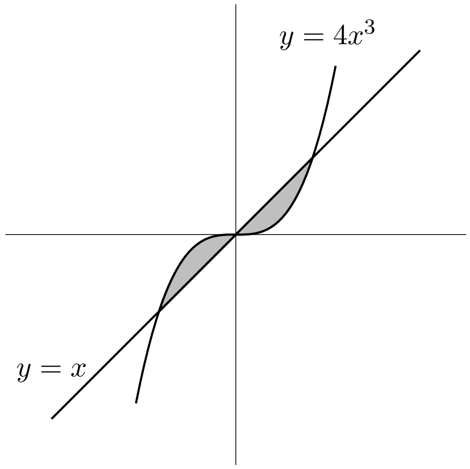
$$= \ln\left(\frac{\sqrt{6}}{2}\right) - \frac{\pi}{12}$$



Return to Problem

(see Math 250, Review Topic 13D for help.)

8.



We must first determine where the curves cross. Set $x = 4x^3$, then $4x^3 - x = 0 \rightarrow x(2x-1)(2x+1) = 0$ so $x = 0, \frac{1}{2}, -\frac{1}{2}$. We can use symmetry and find

$$\text{Area} = 2\int_0^{1/2} (x - 4x^3)dx = 2\left(\frac{x^2}{2} - x^4\right)\Bigg|_0^{1/2}$$

$$2\left[\frac{(\frac{1}{2})^2}{2} - \left(\frac{1}{2}\right)^4 - \left(\frac{0^2 - 0^4}{2}\right)\right] = 2\left[\frac{1}{8} - \frac{1}{16}\right] = \frac{1}{8}$$

Return to Problem

(see Math 250, Review Topic 13D for help.)